Journal of Algebra
Volume 575, 1 June 2021, Pages 220-232

On maps preserving products equal to fixed elements

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https://doi.org/10.1016/j.jalgebra.2021.02.014

Abstract

We obtain a characterization of bijective linear maps $f: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying $f(A)f(B) = M$ whenever $AB = N$, where M and N are fixed $n \times n$ matrices.

Keywords

Maps preserving products; Linear preserver problems; Complex matrices

1. Introduction

Let $M_n = M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices with entries from the complex numbers, for $n \geq 2$. The following question, regarding maps that preserve products equal to fixed elements, was posed in [4]:

Problem 1

If $M, N \in M_n$ are fixed and $f : M_n \to M_n$ is a bijective linear map such that $f(A)f(B) = M$ whenever $AB = N$, then can we find a description of f?

Our goal in this paper is to answer this question.

This problem has been extensively studied in partial forms. Perhaps the most studied case is when $N = M = 0$, and f is familiarly called a zero product preserving map. Characterizations of zero product preserving maps on various algebras, including group algebras, matrices over division rings, prime algebras, and von Neumann algebras, can be found in [1], [2], [8], [11], [16]. Each of these results comes to the same conclusion: the map must be the product of a central element and a homomorphism.

A natural extension of the zero product preserving map is one preserving the identity product, i.e. when $N = 1$. Chebotar, Ke, Lee, and Shiao found that a bijective additive map α on a division ring $\mathscr{D}$ that satisfies $\alpha(a^{-1})\alpha(a) = \alpha(b^{-1})\alpha(b)$ for all nonzero $a, b \in \mathscr{D}$ must have the form $\alpha(x) = \alpha(1)\varphi(x)$, where φ is an automorphism or antiautomorphism, and $\alpha(1)$ is a central element of $\mathscr{D}$ [9]. Shortly thereafter, Lin and Wong generalized this result to $M_n(\mathscr{D})$ [15].

One can also consider extending the identity from Problem 1 to other types of matrix products; that is, describing bijective linear maps satisfying $f(A)_* f(B) = M$ whenever $A_* B = N$. The case when $_*$ represents the Jordan product was described completely by the first author, Hsu, and Kapalko [6]. When $_*$ represents the Lie product, general approaches need to be developed further. See [13] and [14] for low-rank cases.

The next goal in Problem 1 (with the ordinary product) was to study more arbitrary forms for N and M. The first author studied the case when N is invertible. In particular, assuming that α satisfies $\alpha(x)\alpha(y) = m$ whenever $xy = k$ for some fixed m, k (with k nonzero) in a division ring $\mathscr{D}$ [3], she showed α has the form $\alpha(x) = \alpha(1)\varphi(x)$, where φ is an automorphism or antiautomorphism, but $\alpha(1)$ is not necessarily central. Shortly thereafter, this result was generalized in [6] to $M_n(\mathscr{D})$, where $\mathscr{D}$ has characteristic 0, with the assumption that k, m are invertible elements of $M_n(\mathscr{D})$.

At this point, it became clear that there is a difference between the cases where $N = M = 0$ and where N is invertible. Given a map that preserves the zero product, we can anticipate that the map is a homomorphism multiplied by a central element. However, given that N and M are invertible, any map satisfying $f(A)f(B) = M$ whenever $AB = N$ is a Jordan homomorphism multiplied by an element that is not necessarily central. These differences led researchers to consider what would happen when N and M are "between" zero and invertible.

The first investigation concerned preserving products equal to rank-one matrices. The case when N and M are rank-one idempotent matrices was handled in [4], followed by the case when N and M are rank-one nilpotent matrices, studied by the first author and Chang-Lee [5]. In both of these situations, the result was found to resemble the zero product case: the map is a homomorphism multiplied by a scalar.

The final partial result of Problem 1 that we will discuss is when N and M are diagonalizable with the same eigenvalues. This situation, studied by both authors, corroborated the difference between the situations when N and M are invertible and when N and M are noninvertible. In particular, if N and M are invertible diagonalizable matrices, then certainly the description found in [6] holds. However if $\mathrm{rank}(N) = \mathrm{rank}(M) < n$ then f is the product of a central element and a homomorphism [7].

Although many partial cases have been studied, given an arbitrary N, the form of f has not been completely resolved until now. As mentioned, [6] described f in the case when N is invertible, so this paper focuses on describing the form of f when N has rank strictly less than n. In particular, the first theorem confirms the results found in [4], [5] with completely arbitrary M.

Theorem 2

Let $N \in M_n$ be a fixed matrix with $\mathrm{rank}(N) \le n-2$, and let $M \in M_n$ be any fixed matrix. If $f : M_n \to M_n$ is a bijective linear map such that $f(A)f(B) = M$ whenever $AB = N$, then there exists a nonzero $\alpha \in \mathbb{C}$ and invertible $U \in M_n$ such that $f(X) = \alpha UXU^{-1}$ for all $X \in M_n$.

The proof of this theorem, which appears in Section 2, illustrates a technique recently appearing in [5] concerning the so-called "zero-2 pairs" that helps reduce nonzero product preserver problems to the zero product preserver problem. If $\mathrm{rank}(N) = n-1$, there are computational issues with the zero-2 pair method. Consequently, we make an additional, natural assumption in this case that $\mathrm{rank}(M) = \mathrm{rank}(N) = n-1$.

Theorem 3

Let $N, M \in M_n$ be fixed matrices of rank-$(n-1)$. If $f : M_n \to M_n$ is a bijective linear map such that $f(A)f(B) = M$ whenever $AB = N$, then there exists a nonzero $\alpha \in \mathbb{C}$ and invertible $U \in M_n$ such that $f(X) = \alpha UXU^{-1}$ for all $X \in M_n$.

Let J_N and J_M be the Jordan canonical forms of N and M, respectively, and let P, Q be invertible matrices such that $J_N = PNP^{-1}$ and $J_M = QMQ^{-1}$. Let $X' = PXP^{-1}$ for all $X \in M_n$ and define $f'(X') = Qf(X)Q^{-1}$ (which is clearly a bijective linear map). From this, we can see that the property $AB = N$ implies $f(A)f(B) = M$ is equivalent to the property $A'B' = J_N$ implies $f'(A)f'(B) = J_M$. However, f is of the form αUXU^{-1} if and only if f' is of the same form. Therefore, throughout the paper, we will assume that the matrices N and M are in Jordan canonical form.

2. Proof of Theorem 2

Throughout this section, we will assume that $\mathrm{rank}(N) \leq n-2$, M is fixed but arbitrary, and $f : M_n \to M_n$ is a bijective, linear map such that $f(A)f(B) = M$ whenever $AB = N$.

Since N is in Jordan canonical form and $\mathrm{rank}(N) \leq n-2$, N has at least two rows of zeros. First note that if $N = 0$, then $M = 0$ and we recover the zero product preserver problem [16]. In particular, the theorem holds if $n = 2$. Assume going forward that $N \neq 0$.

Write

$$N = \begin{pmatrix} J & 0 \\ 0 & S \end{pmatrix},$$

where J is the direct sum of all Jordan blocks of N corresponding to nonzero eigenvalues and S is the direct sum of all Jordan blocks corresponding to zero eigenvalues. We will also define the size of the block J to be $n_1 \times n_1$ and the size of the block S to be $n_2 \times n_2$, so that $n_1 + n_2 = n$.

Our first goal is to show that the matrix units e_{ij} "behave like" matrix units in the image of f.

Lemma 4

For all $i, j, k, l \in \{1, \ldots, n\}$, we have

(1) $f(e_{ij})f(e_{kl}) = 0$ *whenever $j \neq k$, and*

(2) $f(e_{ij})f(e_{jl}) = f(e_{ik})f(e_{kl})$.

Proof

First, we fix j and k with $j \neq k$. Without loss of generality, we will assume that rows p and q (with $p \neq q$) of N are zero. Let $\sigma \in S_n$ be a permutation such that $\sigma(p) = j$ and $\sigma(q) = k$. We can construct matrices A and B as follows. Set

$$A = e_{1,\sigma(1)} + \ldots + e_{n_1,\sigma(n_1)} + \lambda_{n_1+1} e_{n_1+1,\sigma(n_1+1)} + \lambda_{n-1} e_{n-1,\sigma(n-1)},$$

where λ_i is either 1 (if the $e_{i,i+1}$ entry of N is nonzero) or 0 (if the $e_{i,i+1}$ entry of N is zero). In particular, we have $\lambda_p = \lambda_q = 0$. Then

$$B = J_\sigma + \lambda_{n_1+1} e_{\sigma(n_1+1),n_1+2} + \ldots + \lambda_{n-1} e_{\sigma(n-1),n},$$

where, given $J = (a_{ij} e_{ij})$ for $a_{ij} \in \mathbb{C}$, we define $J_\sigma = (a_{ij} e_{\sigma(i)j})$. We note that j and k will never appear as the second index of any $e_{i,\sigma(i)}$ term of A, nor as the first index of any $e_{\sigma(i),i+1}$ term of B.

Now, choose any indices i and l. We can see that

$$(A + e_{ij})B = A(B + e_{kl}) = (A + e_{ij})(B + e_{kl}) = N,$$

and by assumption, this yields

$$f(A + e_{ij})f(B) = M, \tag{1}$$

$$f(A)f(B + e_{kl}) = M, \text{ and} \tag{2}$$

$$f(A + e_{ij})f(B + e_{kl}) = M. \tag{3}$$

Using the linearity of f and fact that $f(A)f(B) = M$, we can see that equations (1) and (2) give us

$$f(e_{ij})f(B) = 0 \text{ and } f(A)f(e_{kl}) = 0, \tag{4}$$

respectively. As f is linear, we get from (3) and (4) that

$$f(e_{ij})f(e_{kl}) = 0. \tag{5}$$

This proves the first statement.

Next, using a technique similar to what we used in finding (4) (and the same permutation σ), we obtain

$$f(e_{ik})f(B) = 0 \text{ and } f(A)f(e_{jl}) = 0. \tag{6}$$

From $(A + e_{ij} + e_{ik})(B + e_{jl} - e_{kl}) = N$, we have

$$\begin{aligned} M &= f(A + e_{ij} + e_{ik})f(B + e_{jl} - e_{kl}) \\ &= f(A)f(B) + f(A)f(e_{jl}) + f(A)f(e_{kl}) \\ &\quad + f(e_{ij})f(B) + f(e_{ij})f(e_{jl}) - f(e_{ij})f(e_{kl}) \\ &\quad + f(e_{ik})f(B) + f(e_{ik})f(e_{jl}) - f(e_{ik})f(e_{kl}). \end{aligned}$$

The fact that $f(A)f(B) = M$, along with equations (4), (5), and (6), gives us that

$$f(e_{ij})f(e_{jl}) = f(e_{ik})f(e_{kl}), \tag{7}$$

as desired. □

We will now introduce zero-2 pairs. Let $\mathscr{X}$ be the set of all matrices with all rows zero except (possibly) one, and let $\mathscr{Y}$ be the set of all matrices with all columns zero except (possibly) one. Let $\mathscr{X}_2$ be a subset of $\mathscr{X}$ such that each element in $\mathscr{X}_2$ has at most 2 non-zero entries, and analogously, let $\mathscr{Y}_2$ be a subset of $\mathscr{Y}$ such that each element of $\mathscr{Y}_2$ has at most 2 non-zero entries. By a zero-2 pair, we mean a pair $(X_2, Y_2) \in \mathscr{X}_2 \times \mathscr{Y}_2$ such that $X_2 Y_2 = 0$. Additionally, we say that *f* preserves zero products on zero-2 pairs if $f(X_2)f(Y_2) = 0$ for all zero-2 pairs (X_2, Y_2).

By Corollary 5 from [5], any linear map that preserves zero products on zero-2 pairs also preserves zero products. To prove Theorem 2 we will show that *f* preserves zero products on zero-2 pairs, and so it is of the form described by [10].

Proof of Theorem 2

Let (X_2, Y_2) be a zero-2 pair. Without loss of generality, we will assume $X_2 \neq 0$ and $Y_2 \neq 0$. By assumption, $X_2 = c_1 e_{ij} + c_2 e_{il}$ where *j* and *l* are distinct and $c_1, c_2 \in \mathbb{C}$. Similarly $Y_2 = d_1 e_{pk} + d_2 e_{qk}$, where *p* and *q* are distinct and $d_1, d_2 \in \mathbb{C}$.

If we assume that $c_1 = 0$, $c_2 \neq 0$, $d_1 \neq 0$, and $d_2 \neq 0$, then (X_2, Y_2) is a zero-2 pair only if $l \neq p, q$. From this and Lemma 4 part (1), we can see that

$$f(X_2)f(Y_2) = f(c_2 e_{il})f(d_1 e_{pk} + d_2 e_{qk}) = c_2 d_1 f(e_{il})f(e_{pk}) + c_2 d_2 f(e_{il})f(e_{qk}) = 0.$$

The situation is analogous when $c_1 \neq 0$, $c_2 \neq 0$, $d_1 = 0$, and $d_2 \neq 0$, as well as when $c_1 = 0$, $c_2 \neq 0$,

$d_1 = 0$, and $d_2 \neq 0$. Therefore, we are left with the situation that all c_1, c_2, d_1, d_2 are nonzero.

If $p \neq j$ and $p \neq l$, then we can see that $q \neq j$ and $q \neq l$, and we have

$$\begin{aligned}
&f(X_2)f(Y_2) = f(c_1 e_{ij} + c_2 e_{il})f(d_1 e_{pk} + d_2 e_{qk}) \\
&= c_1 d_1 f(e_{ij})f(e_{pk}) + c_1 d_2 f(e_{ij})f(e_{qk}) + c_2 d_1 f(e_{il})f(e_{pk}) + \\
&\ c_2 d_2 f(e_{il})f(e_{qk}) \\
&= 0
\end{aligned}$$

by Lemma 4 part (1).

If $p = j$, then we must have $q = l$, and in this case, we note that $X_2 Y_2 = 0$ implies that $c_1 d_1 + c_2 d_2 = 0$. Then, using Lemma 4,

$$\begin{aligned}
&f(X_2)f(Y_2) = f(c_1 e_{ij} + c_2 e_{il})f(d_1 e_{jk} + d_2 e_{lk}) \\
&= c_1 d_1 f(e_{ij})f(e_{jk}) + c_1 d_2 f(e_{ij})f(e_{lk}) + c_2 d_1 f(e_{il})f(e_{jk}) + \\
&\ c_2 d_2 f(e_{il})f(e_{lk}) \\
&= 2(c_1 d_1 + c_2 d_2)f(e_{ij})f(e_{jk}) \\
&= 0.
\end{aligned}$$

We have considered all possibilities for the zero-2 pair (X_2, Y_2), and have shown that $f(X_2)f(Y_2) = 0$. Hence f preserves the zero product, so there is a nonzero $\alpha \in \mathbb{C}$ and invertible $U \in M_n$ such that $f(X) = \alpha U X U^{-1}$ for all $X \in M_n$. □

3. Proof of Theorem 3

In this section, we assume that M and N are of rank $n-1$, and $f : M_n \to M_n$ is linear such that $f(A)f(B) = M$ whenever $AB = N$.

To use the zero-2 pairs method, one must show that $f(e_{ij})f(e_{kl}) = 0$ whenever $j \neq k$. If $S \in M_n$ is a rank-$(n-1)$ nilpotent, then S is similar to

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Consider then a bijective linear map $g : M_n \to M_n$ that preserves products equal to S. Notice that if $AB = S$ then at least one of A or B must be rank-$(n-1)$. By Lemma 6 below, this means either the last row of A or the first column of B must be zero, but not both. So it is hard to imagine how one could prove directly that $g(e_{n1})g(e_{n1}) = 0$. Hence in this section we set out to handle the rank-$(n-1)$ case using a different approach, with the natural assumption that $\mathrm{rank}(M) = n-1$.

Given integers n_1 and n_2 such that $n_1 + n_2 = n$, let $S \in M_{n_2}$ be the nilpotent matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since $\mathrm{rank}(N) = n-1$, it can be expressed as

$$N = \begin{pmatrix} J & 0 \\ 0 & S \end{pmatrix}$$

for some $J \in M_{n_1}$ invertible already in Jordan form because N is. Without loss of generality we may also insist that the blocks of J are arranged increasing by size. The proof considers both the case when $n_1 > 0$ and the case when $n_1 = 0$ (i.e. when $N = S$).

Lemma 5

If B is a rank-$(n-1)$ matrix whose (n_1+1)th column is zero, there exists a matrix A such that $AB = N$

.

Proof

Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ denote the standard basis vectors of $\mathbb{C}^n$. Since $\mathrm{rank}(B) = n-1$, the vectors $B\vec{e}_1, B\vec{e}_2, \ldots, B\vec{e}_{n_1}, B\vec{e}_{n_1+2}, \ldots, B\vec{e}_n$ are linearly independent with $B\vec{e}_{n_1+1} = 0$. Now, the (n_1+1) th column of N is also zero, while the remaining column vectors are linearly independent. Hence there exists a linear transformation A that maps the $\{B\vec{e}_i\}_{i \neq (n_1+1)}$ to the column vectors of N. Interpreting A as a matrix yields the equation $AB = N$.

This can be done explicitly as follows. Let ℓ be the number of 1×1 Jordan blocks of J (recall the blocks of J are arranged increasing by size) and let λ_j denote the (j,j)-entry of N. If $\ell = 0$, partition the linearly independent set $\{B\vec{e}_i\}_{i \neq (n_1+1)}$ into two linearly independent disjoint subsets

$$\{B\vec{e}_1, \ldots, B\vec{e}_{n_1}\}, \quad \text{and} \quad \{B\vec{e}_{n_1+2}, \ldots, B\vec{e}_n\} \tag{8}$$

and define A to be a linear transformation such that $AB\vec{e}_1 = \lambda_1\vec{e}_1$, $AB\vec{e}_i = \lambda_i\vec{e}_i + \vec{e}_{i-1}$ for $1 < i < n_1$, and $AB\vec{e}_i = \vec{e}_{i-1}$ for $n_1 + 2 \le i \le n$. Thus the transformation A maps the first subset of vectors into the column vectors of J and the second subset of vectors into the column vectors of S. The transformation A, when viewed as an $n \times n$ matrix, satisfies $AB = N$.

If $\ell = n_1$, use the same partition of $\{B\vec{e}_i\}_{i \ne (n_1+1)}$ into the subsets as in (8) and define A to be a linear transformation satisfying $AB\vec{e}_i = \lambda_i\vec{e}_i$ for all $1 \le i \le n_1$ and $AB\vec{e}_i = \vec{e}_{i-1}$ for $n_1 + 2 \le i \le n$. Hence A, for the same reason as above, becomes a matrix satisfying $AB = N$.

If $1 \le \ell < n_1$, partition $\{B\vec{e}_i\}_{i \ne (n_1+1)}$ into three linearly independent disjoint subsets

$$\{B\vec{e}_1, \ldots, B\vec{e}_\ell\}, \quad \{B\vec{e}_{\ell+1}, \ldots, B\vec{e}_{n_1}\}, \quad \text{and} \quad \{B\vec{e}_{n_1+2}, \ldots, B\vec{e}_n\}.$$

Now define A to be a linear transformation such that $AB\vec{e}_i = \lambda_i e_i$ for $1 \le i \le \ell$, $AB\vec{e}_{\ell+1} = \lambda_{\ell+1}\vec{e}_{\ell+1}$, and $AB\vec{e}_i = \lambda\vec{e}_i + \vec{e}_{i-1}$ for $\ell + 1 < i \le n_1$. This means that A maps the first two subsets of vectors into the column vectors of J, and as before, we can also specify that the third subset be mapped to the columns of S. Thus A becomes a matrix satisfying $AB = N$. □

Let $\mathrm{ann}_\ell(T) = \{C \in M_n : CT = 0\}$ and $\mathrm{ann}_r(T) = \{C \in M_n : TC = 0\}$ denote the left and right annihilator of a matrix T, respectively.

Lemma 6

Suppose $AB = T$, *where* $\mathrm{rank}(T) = n - 1$ and T is in *Jordan canonical form*. If $\mathrm{rank}(A) = n - 1$, *then* $\mathrm{ann}_\ell(A) = \mathrm{ann}_\ell(T)$. *If* $\mathrm{rank}(B) = n - 1$, *then* $\mathrm{ann}_r(B) = \mathrm{ann}_r(T)$.

Proof

Suppose $\mathrm{rank}(A) = n - 1$. If $C \in \mathrm{ann}_\ell(A)$ then $0 = CAB = CT$, so $\mathrm{ann}_\ell(A) \subseteq \mathrm{ann}_\ell(T)$. Equality follows since $\dim \mathrm{ann}_\ell(A) = \dim \mathrm{ann}_\ell(T) = n$. Indeed, as $\mathrm{rank}(A) = n - 1$ there exist invertible matrices P and Q such that

$$PAQ = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

If $CA = 0$, then

$$0 = (CP^{-1})(PAQ) = CP^{-1}\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence the first $n-1$ columns of CP^{-1} are identically zero, while the entire nth column can be arbitrary, and so $C = Pe_{jn}$ is a left annihilator of A. However, since P is invertible, we must have e_{jn} is a left annihilator of A for all $j \in \{1, \ldots, n\}$, and so $\dim \mathrm{ann}_\ell(A) = n$ (alternatively one can consider the left annihilators of matrices in Jordan form to obtain the same conclusion). The statement $\mathrm{ann}_r(B) = \mathrm{ann}_r(T)$ is completely analogous. □

Remark 7

From the proof of Lemma 6, one can see that if $AB = N$ and $\mathrm{rank}(A) = n-1$, then A is of the form

$$A = \begin{pmatrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \\ 0 & \cdots & 0 \end{pmatrix}.$$

Likewise if $AB = N$ and $\mathrm{rank}(B) = n-1$, B is of the form

$$B = \begin{pmatrix} * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * \end{pmatrix},$$

where the (n_1+1)th column of B is zero.

The last step before we begin the proof of Theorem 3 is to establish at least a partial version of Lemma 4 adapted to the rank-$(n-1)$ case.

Lemma 8

Let $i, j, k, l \in \{1, \ldots, n\}$ with $j \neq k$. Then $f(e_{1j})f(e_{kl}) = 0$.

Proof

Assume first that $n_1 = 0$, so $N = S$. Let $\sigma \in S_n$ be a permutation. It is easy to see that the matrices

$$A_\sigma = e_{1\sigma(1)} + e_{2\sigma(2)} + \cdots + e_{n-1,\sigma(n-1)}$$

and

$$B_\sigma = e_{\sigma(1)2} + e_{\sigma(2)3} + \cdots + e_{\sigma(n-1),n}$$

satisfy $A_\sigma B_\sigma = N$ for all $\sigma \in S_n$. Choose σ such that $\sigma(1) = j$ and $\sigma(n) = k$. Hence A_σ contains e_{1j} as a term and

$$A_\sigma(B_\sigma + e_{kl}) = N.$$

It follows from the usual linearity argument that

$$f(A_\sigma)f(e_{kl}) = 0. \tag{9}$$

Analogously we define

$$A_\sigma^- = -e_{1\sigma(1)} + e_{2\sigma(2)} + \cdots + e_{n-1,\sigma(n-1)}$$

and

$$B_\sigma^- = -e_{\sigma(1)2} + e_{\sigma(2)3} + \cdots + e_{\sigma(n-1),n}.$$

Let σ be the same permutation as above. Then

$$A_\sigma^-(B_\sigma^- + e_{kl}) = N.$$

Consequently

$$f(A_\sigma^-)f(e_{kl}) = 0. \tag{10}$$

Comparing terms in equations (9) and (10), we obtain $f(e_{1\sigma(1)})f(e_{kl}) = f(e_{1j})f(e_{kl}) = 0$.

The case $n_1 > 0$ parallels the previous argument with some minor differences. Again, let $\sigma \in S_n$ be such that $\sigma(1) = j$ and $\sigma(n) = k$. Define the matrices

$$I_{n_1,\sigma} = e_{1,\sigma(1)} + e_{2,\sigma(2)} + \cdots + e_{n_1,\sigma(n_1)},$$

$$A_\sigma = e_{n_1+1,\sigma(n_1+1)} + e_{n_1+2,\sigma(n_1+2)} + \cdots + e_{n-1,\sigma(n-1)},$$

and

$$B_\sigma = e_{\sigma(n_1+1),n_1+2} + e_{\sigma(n_1+2),n_1+3} + \cdots + e_{\sigma(n-1),n}.$$

Also, assuming that $J = \sum_{i,j} a_{ij}e_{ij}$, we define $J_\sigma = \sum_{i,j} a_{ij}e_{\sigma(i)j}$, so that we have

$$(I_{n_1,\sigma} + A_\sigma)(J_\sigma + B_\sigma) = N$$

and also

$$(I_{n_1,\sigma} + A_\sigma)(J_\sigma + B_\sigma + e_{kl}) = N.$$

By linearity of f, we obtain

$$f(I_{n_1,\sigma} + A_\sigma)f(e_{kl}) = 0. \tag{11}$$

Similarly, if we define

$$I^-_{n_1,\sigma} = -e_{1,\sigma(1)} + e_{2,\sigma(2)} + \cdots + e_{n_1,\sigma(n_1)},$$

and define J^-_σ by adding a negative to both the $e_{\sigma(1)1}$ and $e_{\sigma(1)2}$ terms of J_σ, then

$$(I^-_{n_1,\sigma} + A_\sigma)(J^-_\sigma + B_\sigma + e_{kl}) = N,$$

and using linearity of f, we have

$$f(I^-_{n_1,\sigma} + A_\sigma)f(e_{kl}) = 0. \tag{12}$$

Comparing terms in equations (11) and (12), we obtain the desired result: $f(e_{1\sigma(1)})f(e_{kl}) = f(e_{1j})f(e_{kl}) = 0.$ □

We are now ready to prove Theorem 3.

Proof of Theorem 3

We first show that f preserves invertible matrices. Let B be a matrix such that $\mathrm{rank}(B) = n-1$ and the (n_1+1)th column of B is zero. By Lemma 5 there exists a matrix A such that $AB = N$. Hence $f(A)f(B) = M$.

Either $\mathrm{rank}(f(B)) = n-1$ or $\mathrm{rank}(f(B)) = n$. Suppose the latter. Since $\mathrm{rank}(B) = n-1$, there is a nonzero matrix C such that $CB = 0$. Hence $(A+C)B = N$ implies that

$f(A)f(B) + f(C)f(B) = M$. Canceling $f(A)f(B) = M$, we conclude that $f(B)$ has a left zero-divisor, contradicting invertibility. Hence $\mathrm{rank}(f(B)) = n - 1$. Suppose that M has all zeros in the mth column as promised by Remark 7. Then we have that

$$B = \begin{pmatrix} * & \cdots & * & \overset{n_1+1}{0} & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * \end{pmatrix} \Rightarrow f(B) = \begin{pmatrix} * & \cdots & * & \overset{m}{0} & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * \end{pmatrix} \tag{13}$$

By selecting a basis for the subspace of matrices whose $(n_1 + 1)$th column is zero, it follows that f maps this subspace bijectively into the subspace of all matrices whose mth column is zero. Let K be an invertible matrix. Since $NK^{-1}K = N$, we have

$$f(NK^{-1})f(K) = N.$$

We have either $\mathrm{rank}(f(K)) = n - 1$ or $\mathrm{rank}(f(K)) = n$. Suppose the former. Then the mth column of $f(K)$ is zero by Remark 7. By bijectivity we conclude that K is a linear combination of matrices whose $(n_1 + 1)$th column is zero, contradicting invertibility. Hence $\mathrm{rank}(f(K)) = n$.

Since K can be an arbitrary invertible matrix, it follows that $f(K)$ is invertible whenever K is invertible. By [Theorem 2.1, [12]] there are invertible matrices $U, V \in M_n$ such that $f(X) = UXV$ or $f(X) = UX^TV$ for all $X \in M_n$. We claim that the transpose case is not a possibility for f. Indeed, from Lemma 8, fixing $j \neq k$, we can see that

$$0 = f(e_{1j})f(e_{kl}) = (Ue_{j1}V)(Ue_{lk}V).$$

However, $0 = e_{j1}VUe_{lk}$ implies that the $(1, l)$ entry of VU is zero. Since l is arbitrary, the entire first row of VU has to be zero, contradicting the invertibility of U and V. Therefore, we must have $f(X) = UXV$ for all $X \in M_n$, or equivalently,

$$f(X) = UX(VU)U^{-1} \tag{14}$$

for all $X \in M_n$. It remains to determine VU.

By Lemma 8, as $j \neq k$, we have

$$0 = f(e_{1j})f(e_{kl}) = (Ue_{1j}V)(Ue_{kl}V),$$

and $0 = e_{1j}VUe_{kl}$ implies that the (j,k) entry of VU is zero. Since we may choose any j and k such that $j \neq k$, then VU must be

$$VU = \mathbf{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{n_1}, \alpha_{n_1+1}, \alpha_{n_1+2}, \ldots, \alpha_n). \tag{15}$$

Since $f(N)f(I_n) = f(I_n)f(N)$, we have $NVU = VUN$. Hence VU is a direct sum of scalar matrices in accordance with the direct sum of N as Jordan blocks; in other words, the diagonal entries take on constant values whenever the diagonal entries of N are constant. In particular, $\alpha_{n_1+1} = \alpha_{n_1+2} = \cdots = \alpha_n$. Define $\alpha := \alpha_{n_1+1}$. If $n_1 = 0$ (that is, $N = S$), this implies that $f(X) = \alpha UXU^{-1}$ for all $X \in M_n$, which settles the purely nilpotent case.

Suppose that $n_1 > 0$. Notice first that $e_{ik}VU = \alpha e_{ik}$ implies that $f(e_{ik}) = \alpha Ue_{ik}U^{-1}$ for all $i \in \{1, \ldots, n\}$ and $k \in \{n_1+1, \ldots, n\}$. In the case $n_1 < n-1$ (that is, N has a nonzero nilpotent block), let $Q = e_{n_1+2,n_1+2} + e_{n_1+3,n_1+3} + \cdots + e_{nn}$ (clearly $SQ = S$). If $1 \leq j \leq m$, then

$$\begin{pmatrix} I_{n_1} & e_{j,n_1+1} \\ 0 & S \end{pmatrix} \begin{pmatrix} J & e_{j,n_1+1} \\ 0 & Q - e_{n_1+1,n_1+1} \end{pmatrix} = \begin{pmatrix} J & e_{j,n_1+1} - e_{j,n_1+1} \\ 0 & S \end{pmatrix} = N,$$

so $f(I_{n_1} + S + e_{j,n_1+1})f(J + Q + e_{j,n_1+1} - e_{n_1+1,n_1+1}) = M$. Using the fact that $f(I_{n_1} + S)f(J + Q) = M$, we conclude that

$$f(I_{n_1} + S)f(e_{j,n_1+1} - e_{n_1+1,n_1+1}) + f(e_{j,n_1+1})f(J + Q + e_{j,n_1+1} - e_{n_1+1,n_1+1}) = 0.$$

Using equation (14),

$$(I_{n_1} + S)VU(e_{j,n_1+1} - e_{n_1+1,n_1+1})VU + e_{j,n_1+1}VU(J + Q + e_{j,n_1+1} - e_{n_1+1,n_1+1})VU = 0$$

and so

$$(I_{n_1}VU + \alpha S)(\alpha e_{j,n_1+1} - \alpha e_{n_1+1,n_1+1}) + \alpha e_{j,n_1+1}(JVU + \alpha Q + \alpha e_{j,n_1+1} - \alpha e_{n_1+1,n_1+1}) = 0.$$

Many of the products above are zero and the equation reduces to

$$\alpha I_{n_1}VUe_{j,n_1+1} - \alpha^2 e_{j,n_1+1} = 0.$$

Since $I_{n_1} VUe_{j,n_1+1} = \alpha_j e_{j,n_1+1}$, we conclude that

$$\alpha\alpha_j e_{j,n_1+1} - \alpha^2 e_{j,n_1+1} = 0.$$

Thus $\alpha_j = \alpha$ for all $j \in \{1, \ldots, n_1\}$. In the case $n_1 = n - 1$, the entire argument can be repeated by replacing S and Q with the 1×1 zero matrix to obtain the same conclusion. Hence $f(X) = \alpha UXU^{-1}$ for all $X \in M_n$. □

We point out that the assumption $\mathrm{rank}(M) = n - 1$ was used only once in the implication (13).

Acknowledgement

The authors would like to thank the referee for his/her useful suggestions.

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