ON A CERTAIN FUNCTIONAL IDENTITY INVOLVING INVERSES

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Abstract. Let R be a simple artinian ring with characteristic different from 2 and 3. The purpose of this paper is to describe additive maps f, g satisfying the identity $f(x)x^{-1} + xg(x^{-1}) = 0$ for every invertible $x \in R$.

1. Introduction

The theory of functional identities extended results in polynomial identities, generalized polynomial identities, and even generalized polynomial identities with derivations and (anti)homomorphisms [2]. However, to the best of our knowledge, there were almost no results that extended rational identities. In 1987, Vukman studied the identity $f(x) = -x^2 f(x^{-1})$ on division rings [5], but this was before even the first results of functional identities. Our goal is to present a basic result that shows how rational identities can be extended to functional identities.

We will consider an identity of the form $f(x)x^{-1} + xg(x^{-1}) = 0$. This basic identity was motivated by the functional identity

$$F(x)x = xG(x) \tag{1}$$

which was studied by Brešar [1]. It follows from Brešar's result [1, Corollary 4.9] that the additive maps F and G that satisfy identity (1) on a division ring D must be of the form

$$F(x) = xa + \zeta(x) \text{ and } G(x) = ax + \zeta(x) \tag{2}$$

for all $x \in D$, where $a \in D$ and ζ is an additive map to the center of the division ring. Such solutions are traditionally called the standard solutions [2]. It seems that in our situation, the standard solutions will be more complicated.

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Theorem 1. *Let D be a division ring with characteristic different from 2, and let $f, g : D \to D$ be additive maps satisfying the identity*

$$f(x)x^{-1} + xg(x^{-1}) = 0 \tag{3}$$

for every $x \in D^{\times}$, where $D^{\times}$ is the set of invertible elements of D. Then

$$f(x) = xq + d(x) \text{ and } g(x) = -qx + d(x),$$

where $d : D \to D$ is a derivation and $q \in D$ is a fixed element.

Remark 2. One can see that the main difference between the solutions (2) of the functional identity (1) and the solutions of our identity (3) is that ζ from (2) is a central map, and in our case, d is an arbitrary derivation. This makes us believe that in the case of more general identities involving inverses, the solutions will be standard modulo the vector space of derivations; that is, the solutions will be standard ignoring terms involving derivations.

Theorem 1 will be proved in section 2. We now present an immediate consequence of the theorem.

Corollary 3. *Let D be a division ring with characteristic different from 2. If $f : D \to D$ is any additive map such that it satisfies the identity*

$$f(x)x^{-1} + xf(x^{-1}) = 0 \tag{4}$$

for all $x \in D^{\times}$, then f is a derivation.

Note that the map $f(x) = x$, which is not a derivation, satisfies identity (4) in the case when D is of characteristic 2, so the condition that D does not have characteristic 2 in Corollary 3 cannot be removed.

In this paper, we will also consider a generalization of Theorem 1.

Theorem 4. *Let D be a division ring with characteristic different from 2 and 3, and let $R = M_n(D)$ be a ring of $n \times n$ matrices with $n \geq 2$. Let $f, g : R \to R$ be additive maps satisfying the identity*

$$f(x)x^{-1} + xg(x^{-1}) = 0 \tag{5}$$

for every $x \in R^{\times}$, where $R^{\times}$ is the set of invertible elements of R. Then

$$f(x) = xq + d(x) \text{ and } g(x) = -qx + d(x),$$

where $d : R \to R$ is a derivation and $q \in R$ is a fixed element.

The question regarding whether or not the exclusion of characteristic 3 is necessary in Theorem 4 remains open.

This theorem yields a corollary analogous to Corollary 3.

Corollary 5. *Let D be a division ring with characteristic different from 2 and 3, and let $R = M_n(D)$ be a ring of $n \times n$ matrices with $n \geq 2$. If $f : R \to R$ is any additive map such that it satisfies the identity $f(x)x^{-1} + xf(x^{-1}) = 0$ for all $x \in R^\times$, then f is a derivation.*

It is interesting to note that a similar result for invertible matrices over commutative rings was obtained in the paper by Ge, Li, and Wang [3]. However, their technique was based on the use of orthogonal idempotents, which is not applicable in our case.

Corollary 6. *Let R be a simple artinian ring with characteristic different from 2 and 3. If $f, g : R \to R$ are any additive maps satisfying the identity $f(x)x^{-1} + xg(x^{-1}) = 0$ for all $x \in R^\times$, then*

$$f(x) = xq + d(x) \text{ and } g(x) = -qx + d(x),$$

where $d : R \to R$ is a derivation and $q \in R$ is a fixed element.

2. Proof of Theorem 1

We will apply Hua's Identity, which states that for any nonzero elements a, b of a division ring with $ab \neq 1$, we have

$$a - aba = \big(a^{-1} + (b^{-1} - a)^{-1}\big)^{-1}. \tag{6}$$

Additionally, we will make use of the following equivalent forms of (3):

$$f(x) = -xg(x^{-1})x \tag{7}$$

and

$$g(x^{-1}) = -x^{-1}f(x)x^{-1}. \tag{8}$$

Let us define $c = a - aba$ for some $a, b \in D^\times$. Now, using (7) with $x = c$, $x^{-1} = a^{-1} + (b^{-1} - a)^{-1}$, and the additivity of g, we see that

$$\begin{aligned} f(c) &= -cg\big(a^{-1} + (b^{-1} - a)^{-1}\big)c \\ &= -cg(a^{-1})c - cg\big((b^{-1} - a)^{-1}\big)c. \end{aligned}$$

Removing g from the equation by applying (8), we have

$$f(c) = ca^{-1}f(a)a^{-1}c + c(b^{-1} - a)^{-1}f(b^{-1} - a)(b^{-1} - a)^{-1}c.$$

Rearranging (6), we can substitute both $(b^{-1}-a)^{-1} = c^{-1}-a^{-1}$ and $c = a-aba$ to achieve

$$f(a - aba) = f(a) - f(a)ba - abf(a) + abf(a)ba + abf(b^{-1} - a)ba.$$

Now, using the additivity of f and simplifying appropriately, we have

$$f(aba) = f(a)ba + abf(a) - abf(b^{-1})ba.$$

One final replacement using (8) yields

$$f(aba) = f(a)ba + abf(a) + ag(b)a. \tag{9}$$

Substituting (7) and (8) into (9), replacing a with a^{-1} and b with b^{-1}, and simplifying appropriately, we similarly conclude

$$g(aba) = g(a)ba + abg(a) + af(b)a.$$

Alternately letting $a = 1$, $b = x$ and $a = x$, $b = 1$, and using the fact that $f(1) = -g(1)$, gives the identities

$$f(x) = xf(1) + f(1)x + g(x), \tag{10}$$

$$g(x) = xg(1) + g(1)x + f(x), \tag{11}$$

$$f(x^2) = xf(x) + f(x)x - xf(1)x, \tag{12}$$

and

$$g(x^2) = xg(x) + g(x)x - xg(1)x. \tag{13}$$

Our next step is to define $h = f + g$. The additivity of f and g immediately yields the additivity of h. Additionally, summing (12) and (13) we get

$$h(x^2) = xh(x) + h(x)x;$$

that is, h is a Jordan derivation of D. A well-known result by Herstein [4, Theorem 3.1] gives us that h is an ordinary derivation of D.

Now, adding $f(x)$ to both sides of (10), we can see that

$$2f(x) = 2xf(1) + [f(1), x] + h(x).$$

Finally, defining the derivation $d : D \to D$ by $2d(x) = [f(1), x] + h(x)$, we achieve

$$f(x) = xq + d(x),$$

where $q = f(1)$. Using a similar argument with (11) yields

$$g(x) = -qx + d(x),$$

and so we have the desired result.

3. Proof of Theorem 4

We begin with a lemma.

Lemma 7. *Let R be a unital ring and assume R contains the elements 2, 3 and their inverses. Let $S = \{x \in R \mid x \text{ and } x + c \text{ are invertible for } c = 1, 2, \text{ or } 3\}$. Then, for additive maps $f, g : R \to R$ satisfying property*

$$f(x)x^{-1} + xg(x^{-1}) = 0$$

for every $x \in R^{\times}$, the map $h := f + g$ satisfies

$$h(x^2) = xh(x) + h(x)x \tag{14}$$

for every $x \in S$.

Proof. Given x and $x + c$ as in the statement of the lemma, we note that $x^{-1} - (x + c)^{-1} = cx^{-1}(x + c)^{-1}$, which implies that

$$\left(x^{-1} - (x + c)^{-1}\right)^{-1} = c^{-1}x^2 + x. \tag{15}$$

Since f is additive, we know $f(a - b) = f(a) - f(b)$ for any $a, b \in R$. Now, assuming that a and b are both invertible elements of R and using (7), which is the equivalent form of the property assumed in the lemma, we can see that

$$(a - b)g\big((a - b)^{-1}\big)(a - b) = ag(a^{-1})a - bg(b^{-1})b.$$

Then, multiplying through on both sides by $(a - b)^{-1}$ yields

$$g\big((a - b)^{-1}\big) = (a - b)^{-1}ag(a^{-1})a(a - b)^{-1} - (a - b)^{-1}bg(b^{-1})b(a - b)^{-1}.$$

Substituting $a = x^{-1}$, $b = (x + c)^{-1}$, and using (15) gives us

$$g\big(c^{-1}x^2 + x\big) = (c^{-1}x + 1)g(x)(c^{-1}x + 1) - c^{-1}xg(x + c)c^{-1}x.$$

Using the additivity of g, we then have

$$\begin{aligned} c^{-1}g(x^2) + g(x) = {} & c^{-2}xg(x)x + c^{-1}xg(x) + c^{-1}g(x)x + g(x) \\ & - c^{-2}xg(x)x - c^{-1}xg(1)x, \end{aligned}$$

and simplifying gives us identity (13). Replacing x with x^{-1} and using (8) gives us identity (12).

Defining $h = f + g$ and summing (12) and (13), we can see that

$$h(x^2) = xh(x) + h(x)x + xh(1)x.$$

However, letting $x = 1$ yields that $h(1) = 3h(1)$ which implies that $2h(1) = 0$. Since R contains 2^{-1}, we must have $h(1) = 0$. Therefore,

$$h(x^2) = xh(x) + h(x)x,$$

as desired. □

Now, we proceed with the proof of Theorem 4. We let D be a division ring and $R = M_n(D)$, and let $f, g : R \to R$ be as in Theorem 4. We define $a_{ij} \in R$ to be such that the i, j entry is an invertible element a of D and all other entries are zero.

At this point, we observe that at least three of $I + a_{ij}$, $2I + a_{ij}$, $3I + a_{ij}$, $4I + a_{ij}$ are invertible. Indeed, if $c_0 I + a_{ij}$ is not invertible for $c_0 \in \{1, 2, 3, 4\}$, then we must have $\det(c_0 I + a_{ij}) = 0$, where by 'det' we mean the Dieudonné determinant. Since there is at most one nonzero entry that does not occur along the main diagonal, we know $\det(c_0 I + a_{ij})$ is exactly the product of the elements along the main diagonal of $c_0 I + a_{ij}$. Therefore, $\det(c_0 I + a_{ij}) = 0$ implies one of the diagonal entries of $c_0 I + a_{ij}$ is zero; that is, $i = j$ and $c_0 + a = 0$. Now if $c \in \{1, 2, 3, 4\}$ is different from c_0, then we have $c + a \neq 0$, and thus, we have $\det(cI + a_{ij}) \neq 0$; that is, $cI + a_{ij}$ is invertible for every $c \in \{1, 2, 3, 4\} \setminus \{c_0\}$, as desired.

Choose $cI + a_{ij}$ with $c \in \{1, 2, 3\}$ such that $cI + a_{ij}$ is invertible. Note that from our observation, we know $(c + c')I + a_{ij}$ is also invertible for $c' \in \{1, 2, 3\}$. Using Lemma 7, we get

$$h\big((cI + a_{ij})^2\big) = (cI + a_{ij})h(cI + a_{ij}) + h(cI + a_{ij})(cI + a_{ij}).$$

Then, the additivity of h implies that

$$\begin{aligned} h\big((cI + a_{ij})^2\big) &= 2c^2 h(I) + 2ch(a_{ij}) + a_{ij}ch(I) + ch(I)a_{ij} \\ &\quad + a_{ij}h(a_{ij}) + h(a_{ij})a_{ij} \\ &= 2ch(a_{ij}) + a_{ij}h(a_{ij}) + h(a_{ij})a_{ij}. \end{aligned}$$

However, the left hand side of the previous equality can also be viewed like so:

$$\begin{aligned} h\big((cI + a_{ij})^2\big) &= h(c^2 I) + 2h(ca_{ij}) + h(a_{ij}^2) \\ &= 2ch(a_{ij}) + h(a_{ij}^2). \end{aligned}$$

Equating the two, we have

$$h(a_{ij}^2) = a_{ij}h(a_{ij}) + h(a_{ij})a_{ij}. \qquad (16)$$

We now observe that at least two of $I+a_{ij}+b_{kl}$, $2I+a_{ij}+b_{kl}$, $3I+a_{ij}+b_{kl}$, $4I+a_{ij}+b_{kl}$ are invertible. Indeed, assume that $c_0I+a_{ij}+b_{kl}$ is not invertible for $c_0 \in \{1,2,3,4\}$. Let $a \in D^\times$ be the i,j entry of a_{ij} and let $b \in D^\times$ be the k,l entry of b_{kl}. There are a few cases that can occur (throughout these cases, we assume $c \in \{1,2,3,4\} \setminus \{c_0\}$).

- Case 1: $i=j=k=l$. In this case, we can see that $c_0I+a_{ij}+b_{kl} = c_0I+a_{ii}+b_{ii}$, so that $\det(c_0I+a_{ii}+b_{ii}) = c_0^{n-1}(c_0+a+b) = 0$, which implies that $c_0 = -(a+b)$. However, $\det(cI+a_{ii}+b_{ii}) \neq 0$, and so $cI+a_{ii}+b_{ii}$ is invertible for three values of c.
- Case 2: $i=j$, $k \neq l$. Here, since the k,l entry is the only nonzero entry outside of the main diagonal, we know $\det(c_0I+a_{ii}+b_{kl}) = c_0^{n-1}(c_0+a) = 0$ and hence, we must have $c_0 = -a$. Again, we have $\det(cI+a_{ii}+b_{kk}) \neq 0$, and so $cI+a_{ii}+b_{kk}$ is invertible for three values of c.
- Case 3: $i=j$, $k=l$, $i \neq k$. In this case, $\det(c_0I+a_{ii}+b_{kk})$ equals $c_0^{n-2}(c_0+a)(c_0+b)$ or $c_0^{n-2}(c_0+b)(c_0+a)$. Either way, this implies that $c_0=-a$ or $c_0=-b$. Without loss of generality assume $c_0=-a$. Then we have $cI+a_{ii}+b_{kk}$ is invertible for $c \neq -b$; that is, $cI+a_{ii}+b_{kk}$ is invertible for at least two values of c.
- Case 4: $i \neq j$, $k \neq l$. Given $\det(c_0I+a_{ij}+b_{kl}) = 0$, we must have that $i=l$, $j=k$, in which case, $\det(c_0I+a_{ij}+b_{ji})$ equals $c_0^{n-2}(c_0^2+(-1)^{i+j}ab)$ or $c_0^{n-2}(c_0^2+(-1)^{i+j}ba)$. This implies that c_0^2 equals $-(-1)^{i+j}ab$ or $-(-1)^{i+j}ba$. If the characteristic of D is 5 or 7, we have that $1^2=4^2$ or $3^2=4^2$, respectively, which implies that $cI+a_{ij}+b_{ji}$ is invertible for at least two values of c. For any other characteristic, we have that $cI+a_{ij}+b_{ji}$ is invertible for three values of c.

In any case, we can see that at least two of $I+a_{ij}+b_{kl}$, $2I+a_{ij}+b_{kl}$, $3I+a_{ij}+b_{kl}$, $4I+a_{ij}+b_{kl}$ are invertible.

Let $cI+a_{ij}+b_{kl}$ be invertible for $c \in \{1,2,3\}$. From our observation, we know $(c+c')I+a_{ij}+b_{kl}$ is also invertible for $c' \in \{1,2,3\}$. Applying Lemma 7, we can see that

$$h\big((cI+a_{ij}+b_{kl})^2\big) = (cI+a_{ij}+b_{kl})h(cI+a_{ij}+b_{kl}) \\ + h(cI+a_{ij}+b_{kl})(cI+a_{ij}+b_{kl}).$$

Using the additivity of h and the fact that $h(I) = 0$, we have

$$\begin{aligned} h\big((cI + a_{ij} + b_{kl})^2\big) &= 2ch(a_{ij}) + 2ch(b_{kl}) + a_{ij}h(a_{ij}) + h(a_{ij})a_{ij} + b_{kl}h(b_{kl}) \\ &\quad + h(b_{kl})b_{kl} + a_{ij}h(b_{kl}) + h(b_{kl})a_{ij} + b_{kl}h(a_{ij}) + h(a_{ij})b_{kl} \\ &= 2ch(a_{ij}) + 2ch(b_{kl}) + h(a_{ij}^2) + h(b_{kl}^2) \\ &\quad + a_{ij}h(b_{kl}) + h(b_{kl})a_{ij} + b_{kl}h(a_{ij}) + h(a_{ij})b_{kl}, \end{aligned}$$

where the second equality uses (16). On the other hand, we can see that

$$\begin{aligned} h\big((cI + a_{ij} + b_{kl})^2\big) &= h(c^2I + 2ca_{ij} + 2cb_{kl} + a_{ij}^2 + b_{kl}^2 + a_{ij}b_{kl} + b_{kl}a_{ij}) \\ &= 2ch(a_{ij}) + 2ch(b_{kl}) + h(a_{ij}^2) + h(b_{kl}^2) \\ &\quad + h(a_{ij}b_{kl}) + h(b_{kl}a_{ij}). \end{aligned}$$

Equating the two expressions for $h\big((cI + a_{ij} + b_{kl})^2\big)$ and simplifying yields

$$h(a_{ij}b_{kl} + b_{kl}a_{ij}) = a_{ij}h(b_{kl}) + h(b_{kl})a_{ij} + b_{kl}h(a_{ij}) + h(a_{ij})b_{kl}; \qquad (17)$$

that is, h is a Jordan derivation. Using the result by Herstein [4, Theorem 3.1], we can see that h is a derivation, and the rest of the proof follows just as the proof of Theorem 1.

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References

[1] Brešar, M. *On generalized biderivations and related maps*, J. Algebra, **172** (1995), 764–786

[2] Brešar, M.; Chebotar, M. A.; Martindale, W. S. III. Functional Identities, Frontiers in Mathematics. Birkhauser Verlag, 2007

[3] Ge, H.; Li, X.; Wang, D. *Maps determined by action on identity-product elements*, Linear Algebra Appl., **436** (2012), 112–119

[4] Herstein I.N. *Jordan derivations of prime rings*, Proc. Amer. Math. Soc., **8** (1957), 1104–1110

[5] Vukman, J. *A note on additive mappings in noncommutative fields*, Bull. Austral. Math. Soc., **36** (1987), 499–502

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